Solving Recurrences

Detailed Example 2

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Solve for $n = 2^k$ using backward substitutions. Show at least 3.

$$x(n) = x(n/2) + n, \quad x(1) = 1$$

Solution: Substitutions build a series by expanding a partial sum. Each substitution adds a term to the series. To see the summation unfold, do not collect the terms or simplify.

$$
\begin{aligned}
x(n) &= \underbrace{x\left(\frac{n}{2}\right)} + n && \text{replace } n \text{ by } n/2 \\
&= \overbrace{\underbrace{x\left(\frac{n}{2^2}\right)} + \frac{n}{2}} \quad + n && \text{subst 1} \\
&= \overbrace{\underbrace{x\left(\frac{n}{2^3}\right)} + \frac{n}{2^2}} \quad + \frac{n}{2} + n && \text{subst 2} \\
&= \overbrace{x\left(\frac{n}{2^4}\right) + \frac{n}{2^3}} \quad + \frac{n}{2^2} + \frac{n}{2^1} + n && \text{subst 3 (notice powers)} \\
&= \cdots
\end{aligned}
$$

Note the pattern: $x\left(\frac{n}{2^i}\right)$ followed by $\frac{n}{2^{i-1}}$

To drop to $x(1)$, set $i = k$ since $2^k = n$

$$
\begin{aligned}
&= \left(\frac{n}{2^k = n}\right) + \frac{n}{2^{k-1}} + \frac{n}{2^{k-2}} + \cdots + \frac{n}{2^2} + \frac{n}{2^1} + n && n \text{ can be written } n/2^0 \\
&= x(1) + n\left(\frac{1}{2^{k-1}} + \cdots + \frac{1}{2^2} + \frac{1}{2} + 1\right) && \text{check useful summations} \\
&= 1 + n\sum_{i=0}^{k-1} \frac{1}{2^i} = 1 + n\left(2 - \frac{1}{2^{k-1}}\right) = 2n - 1 && \text{notice } 1/2^{k-1} = 2/2^k = 2/n
\end{aligned}
$$

Therefore, $2n - 1$ coupled with $n > 0$ is the solution to the recurrence relation $x(n) = x(n/2) + n$ with the initial condition $x(1) = 1$ when n is power of 2 (i.e., $n = 2^k$). It is also the generic term for a family of sequences specified by the recurrence alone. The initial condition specifies a particular one in that family.

Exercise: Repeat for $x(n) = x(n/2) + (n - 1)$, $x(1) = 1$. Write the sequence, check results in *WolframAlpha*.