Solving Recurrences

Step-by-Step First Example

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Solve the recurrence using backward substitutions. Show at least 3 substitutions.

$$x(n) = x(n-1) + n, \quad x(0) = 0$$

Solution: Substitutions build a series. Each substitution adds a term to the series. To see the series unfold, do not collect terms or simplify.

$$x(n) = \underbrace{x(n-1)} \quad + n \qquad \text{replace term by previous + the term's position}$$

$$= \overbrace{\underbrace{x(n-2)} + (n-1)} \quad + n \qquad \text{subst 1}$$

$$= \overbrace{\underbrace{x(n-3)} + (n-2)} \quad + (n-1) + n \qquad \text{subst 2}$$

$$= \overbrace{x(n-\underset{\uparrow}{4}) + (n-\underset{\uparrow}{3})} \quad + (n-2) + (n-1) + n \qquad \text{subst 3}$$

$$= \cdots$$

note $x(n-i)$, $n-(i-1)$ pattern with each subst;

set $i = n$ to drop to $x(0)$

$$= x(n-n) + n-(n-1) \quad + \quad \cdots + (n-1) + n \qquad \text{note tail terms from above}$$

$$= x(0) + 1 \quad + \quad \cdots + (n-1) + n \qquad \text{recall } x(0) = 0$$

$$= 1 + 2 + 3 + \cdots + n$$

$$= \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \text{must set } n \geq 0 \text{ to satisfy init cond}$$

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