cs223fig28.cdr
Wednesday, April 13, 2022 1:03:44 PM
Color profile: Disabled

Composite Default screen

Motivation

Basic computing methods and strategies for **systems of linear equations** and related operations, interesting scenarios and resulting efficiencies.

➪ Take home for this course

➪ Not a math course

Focus algorithmic, computational concerns

Systems of Linear Equations

- Coefficients matrix
- Right-hand side vector

$$\begin{aligned} 2x_1 - x_2 + x_3 &= 1 \\ 4x_1 + x_2 - x_3 &= 5 \\ x_1 + x_2 + x_3 &= 0 \end{aligned}$$

$$\begin{pmatrix} 2 & -1 & 1 \\ 4 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}$$

$$\mathbf{A}x = \mathbf{b}$$

Systems of Linear Equations
A Simpler Instance

- ➪ **Upper matrix**
- ➪ **Back substitution**

simpler instance ≡ easier to solve

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$a_{11}\boldsymbol{x_1} + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{22}\boldsymbol{x_2} + a_{23}x_3 = b_2$$
$$a_{33}\boldsymbol{x_3} = b_3$$

Quiz
Select a suitable basic operation for back substitution. What's the efficiency of the first step? What's the overall efficiency?

Systems of Linear Equations
Instance Simplification

Exercise
Solve using the simpler form, verify solution in the original system in *WolframAlpha*.

Can we transform a general instance to the simpler form?

$$\begin{pmatrix} 2 & -1 & 1 \\ 4 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}$$

$$\Downarrow \qquad \Downarrow$$

$$\begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

Exercise
Make up a system, use *WolframAlpha* to solve, reduce then verify the original solution in the reduced form.

Instance Simplification
Elementary Operations

➪ Augmented matrix

- ✎ Exchange rows
- ✎ Replace r_i by $mr_i,\ m \neq 0$
- ▶ ✎ Replace r_j by $r_j \pm mr_i,\ i \neq j$

$$\begin{aligned} 2x_1 - x_2 + x_3 &= 1 \\ 4x_1 + x_2 - x_3 &= 5 \\ x_1 + x_2 + x_3 &= 0 \end{aligned}$$

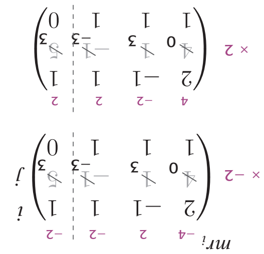
$\times\, m$

$$j \left(\begin{array}{ccc:c} 2 & -1 & 1 & 1 \\ 4 & 1 & -1 & 5 \\ 1 & 1 & 1 & 0 \end{array}\right)$$

Exchange equation with sum/difference of itself and a multiple of another.

Exercise
Eliminate circled element using the first row, what's the multiplier *m*? Use sum of row multiple.

$$\left(\begin{array}{ccc:c} 2 & -1 & 1 & 1 \\ 0 & 3 & -3 & 3 \\ 1 & 1 & 1 & 0 \end{array}\right)$$



Instance Simplification
Gaussian Elimination

- Pivot element
- Multiplier

$$\times -2 \quad \left(\begin{array}{ccc:c} \mathbf{2} & -1 & 1 & 1 \\ 4 & 1 & -1 & 5 \\ 1 & 1 & 1 & 0 \end{array}\right)$$

- Exchange rows
- Replace r_i by mr_i, $m \neq 0$
- Replace r_j by $r_j \pm mr_i$, $i \neq j$

Quiz
Determine a formula for **multipliers** assuming sum of row multiple, use for indicated elements. Modify for difference.

$$\times ? \quad \left(\begin{array}{ccc:c} \mathbf{2} & -1 & 1 & 1 \\ 0 & 3 & -3 & 3 \\ 1 & 1 & 1 & 0 \end{array}\right)$$

Exercise
Develop a pseudocode to describe these procedures.

$$\times ? \quad \left(\begin{array}{ccc:c} 2 & -1 & 1 & 1 \\ 0 & \mathbf{3} & -3 & 3 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right)$$

$$\left(\begin{array}{ccc:c} 2 & -1 & 1 & 1 \\ 0 & 3 & -3 & 3 \\ 0 & 0 & 2 & -2 \end{array}\right)$$

cs223fig28.cdr
Wednesday, April 13, 2022 1:03:55 PM
Color profile: Disabled

Composite Default screen

Gaussian Elimination
A Basic Procedure

➪ Pivot row

➪ Reduced row

3 index **iterators** to: 1) *i*, pick a pivot, 2) *j*, eliminate column elements under pivot, and 3) *k*, adjust (**reduce**) the row for eliminated column element.

$$m_1 = \frac{a_{21}}{a_{11}}$$

$$m_2 = \frac{a_{31}}{a_{11}}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 4 & 1 & -1 & 5 \\ \not{1}\,0 & 1 & 1 & 0 \end{array}\right)$$

$$m_3 = \frac{a_{32}}{a_{22}}$$

$$a_{jk} \leftarrow a_{jk} - a_{ik}\frac{a_{ji}}{a_{ii}}$$

?

Replace r_j by $r_j \pm m r_i$

Quiz
Determine the limits of ***i***, ***j***, ***k*** before you check the algorithm next. *Hint: first index row, second column (fix 1st + vary 2nd to process row).*

Gaussian Elimination
A Straight Procedure

Basic version of a family of techniques originating and refined by many scientists.

Algorithm *GaussElimination*
Input Coefficients matrix $A[1..n, 1..n]$, verctor $b[1..n]$
Output Reduced augmented A, equivalent upper matrix in-place

```
for i ← 1 to n do A[i, n+1] ← b[i]
for i ← 1 to n−1 do                              ▷ pick pivot i
    for j ← i+1 to n do                          ▷ eliminate elements under pivot
        for k ← i to n+1 do                      ▷ adjust/reduce row
            A[j, k] ← A[j, k] − A[i, k] * A[j, i]/A[i, i]
```

Note minor optimization, will it improve efficiency?

Efficiency?