Counting Review
Important Questions

Quiz
Why did bubble sort, selection sort, brute force closest-pair of points have exactly the same number of basic ops, which is the same number of line segments to test in the brute-force convex hull?

3 combinatorial questions for any set of n items

❶ How many ways to order?

❷ How many pairs (ways to pair)?

❸ How many distinct pairs?

A Combinatorial Problem

Cost	Job 1	Job 2	Job 3	Job 4
Person 1	9	2	7	8
Person 2	6	4	3	7
Person 3	5	8	1	8
Person 4	7	6	9	4

Cheaper to assign Job 3 to Person 2, but cheapest Job 3 is really with Person 3

Assign jobs to persons such that total cost is minimum

Assignment Problem

Classic formulation (statement)

- Persons (agents), n
- Jobs (tasks), n
- Cost (weight) function + constraint

Possible objectives: minimize financial cost or completion time, maximize satisfaction.

Applications

- Assign aircrafts to trips
- Equipment to facility, salesman to region…

Minimize trip time or distance, fuel cost or consumption.

Maximize utilization, profit, or productivity, or minimize operating cost.

A brute force approach

In other words, *exhaust* all assignment possibilities.

- ✎ Consider all possible assignments
- ✎ Calculate cost for each assignment
- ✎ Pick one with minimum cost

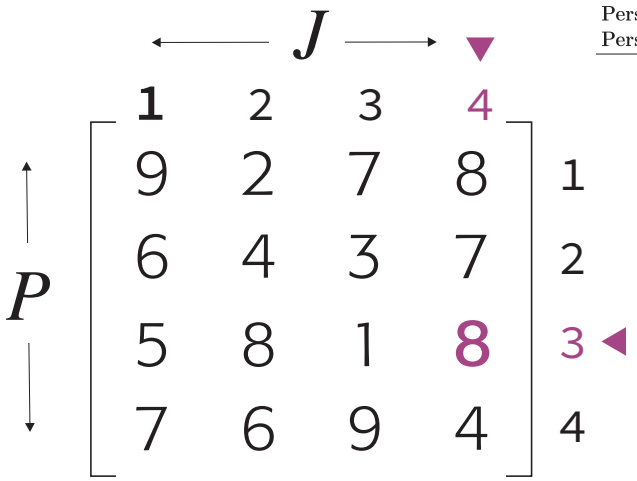
Assignment Problem
Cost Matrix

Cost	Job 1	Job 2	Job 3	Job 4
Person 1	9	2	7	8
Person 2	6	4	3	7
Person 3	5	8	1	**8**
Person 4	7	6	9	4

Algorithm described in terms of a matrix (2D array) organized similar to the original cost table.

Quiz
What's the cost of assigning job 2 to person 4? Write the array element c[?][?].

Note the trivial assignment (what's its cost?)

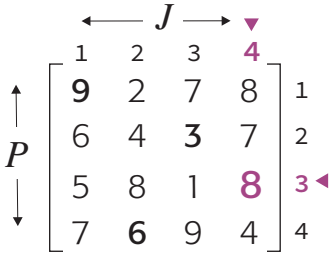


$$\begin{bmatrix} \mathbf{9} & 2 & 7 & 8 \\ 6 & \mathbf{4} & 3 & 7 \\ 5 & 8 & \mathbf{1} & 8 \\ 7 & 6 & 9 & \mathbf{4} \end{bmatrix}$$

Exhaustive Search
Optimal Assignment

➪ Combinatorial objects

Person number index a **job vector** that encodes a job assignment.



Quiz
Can we identify an optimal assignment from the figure?

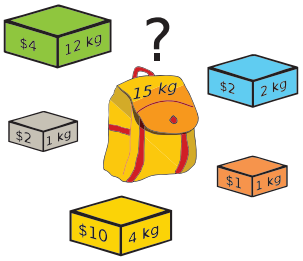
P_1 P_2 P_3 P_4	$\begin{bmatrix}9&2&7&8\\6&4&3&7\\5&8&1&8\\7&6&9&4\end{bmatrix}$	Cost
<1,2,3,4>	9+4+1+4	= 18?
<1,2,4,3>	9+4+8+9	= 30
<1,3,2,4>	9+3+8+4	= 24
<1,3,4,2>	9?+3?+8+6	= 26 ◀
<1,4,2,3>	9+7+8+9	= 33
<1,4,3,2>	9+7+1+6	= 23
...	...	...

job assigned to person 3

Quiz
What are the **combinatorial objects** in this case?

Efficiency?

Knapsack Problem



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$$W, \{w_i\}, \{v_i\}$$

$$\begin{cases} W = 10 \\ w = \{7, 3, 4, 5\} \\ v = \{\$42, \$12, \$40, \$25\} \end{cases}$$

A knapsack in the real world can be a shipping container, a transport plane, or warehouse.

Classic formulation (statement)

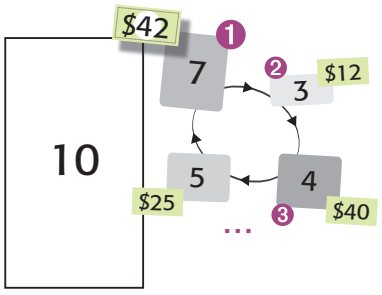
Fit the most valuable set of items without exceeding knapsack capacity

Knapsack Problem
Brute Force Approach

Exhaustive search

- ✎ Consider all possible item subsets
- ✎ Calculate weight for each
- ✎ Pick a highest-value subset that fits

Exhaustive Search
An Optimal Subset



item index set
{1,2,3,4}

Subset	Weight	Value
ϕ	0	0
$\{1\}$	7	42
$\{2\}$	3	12
$\{3\}$	4	40
$\{4\}$	5	25
$\{1,2\}$	10	54
$\{1,3\}$	11	–
$\{1,4\}$	12	–
$\{2,3\}$	7	52
$\{2,4\}$	8	37
$\{3,4\}$	9	65
$\{1,2,3\}$	14	–
$\{1,2,4\}$	15	–
$\{1,3,4\}$	16	–
$\{2,3,4\}$	12	–
$\{1,2,3,4\}$	19	–

Exercise
Identify the optimal subset.

Quiz
Which counting formula is used to count subsets of a certain size? Write a sum for total number of subsets.

Efficiency

Traveling Salesman Problem

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	a	b	c	d	e
a	–	3	1	5	8
b	3	–	6	7	9
c	1	6	–	4	2
d	5	7	4	–	3
e	8	9	2	3	–

Exercise
Specify the formal inputs to the TSP, and give 1-2 instances.

Classic formulation (statement)

Shortest tour through a set of cities visiting each exactly once before returning to the start city

Traveling Salesman Problem
Brute Force Approach

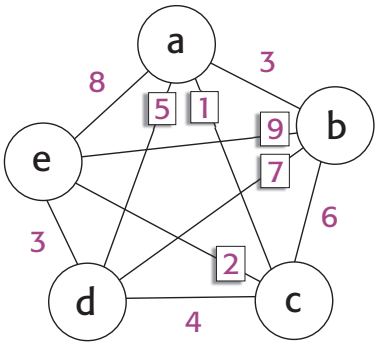
Exhaustive search

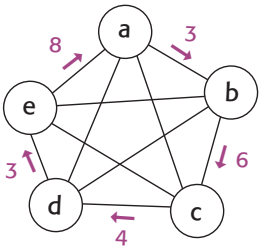
- ✎ Consider all possible tours
- ✎ Calculate the length of each tour
- ✎ Pick one with minimum length

Traveling Salesman Problem
Modeling

➪ Hamiltonian circuit

	a	b	c	d	e
a	–	3	1	5	8
b	3	–	6	7	9
c	1	6	–	4	2
d	5	7	4	–	3
e	8	9	2	3	–





Quiz
What's the length of tour defined by this HC?

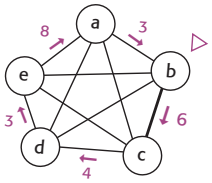
Exercise
Lookup other formulations of (ways to state) the TSP. Explain at least 2. Share in course group.

Find the best HC

- ✎ Min total weights for shortest tour
- ✎ Largest profit? pick max total weights

Hamiltonian Circuits

3 Observations



Exercise ❶
Check the tour length of **abcdea** above if it starts at **b**; write the resulting HC?

Quiz
What are the combinatorial objects in this case?

Quiz
How many HC start/end at **a**?

	1 ▽ a	2 ▽ b	3 ▽ c	4 ▽ d	5 ▽ e
a	–	3	1	5	8
b	3	–	6	7	9
c	1	6	–	4	2
d	5	7	4	–	3
e	8	9	2	3	–

Exercise
Write HC starting with **b**, compare to figure. (Try a few, match with ones in figure).

n+1 vertices (n−1 distinct) ❷

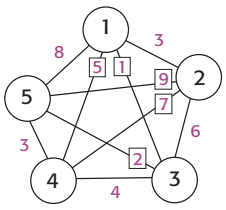
	1,2,3,4,5,1	24	1,5,2,4,3,1	29
▶	1,**3**,**2**,4,5,1 ✕	25	1,2,5,4,3,1	20
	1,4,2,3,5,1	28	1,4,5,2,3,1	24
	1,2,4,3,5,1	24	1,5,4,2,3,1	25
	1,3,4,2,5,1	29	1,2,4,5,3,1	16
	1,4,3,2,5,1	32	1,4,2,5,3,1	24
	1,4,3,5,2,1	23	1,3,2,5,4,1	24
	1,3,4,5,2,1	20	1,2,3,5,4,1	19
	1,5,4,3,2,1	24	1,5,3,2,4,1	28
	1,4,5,3,2,1	19	1,3,5,2,4,1	24
	1,3,5,4,2,1	16	1,2,5,3,4,1	23
	1,5,3,4,2,1	24	1,5,2,3,4,1	32

Exercise ❸
Identify the tour of opposite direction to indicated one.

Exhaustive Search
An Optimal Tour

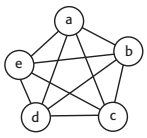
Essentially, only check distinct tours by eliminating opposite and alternate start-verts versions.

	a	b	c	d	e
a	–	3	1	5	8
b	3	–	6	7	9
c	1	6	–	4	2
d	5	7	4	–	3
e	8	9	2	3	–



Exercise
Write the component distances of the first 3 tours.

Exercise
Highlight the optimal HC, show weights.



1,2,3,4,5,1	24	?
1,4,2,3,5,1	28	?
1,2,4,3,5,1	24	?
1,5,2,4,3,1	29	
1,2,5,4,3,1	20	
1,4,5,2,3,1	24	
1,5,4,2,3,1	25	
▶ 1,2,4,5,3,1	16	
1,4,2,5,3,1	24	
1,2,3,5,4,1	19	
1,2,5,3,4,1	23	
1,5,2,3,4,1	32	

Efficiency?

Exercise
How many tours need to be checked if we double cities to 10? How much did it increase?

Exhaustive Search
Performance Summary

➪ **Results summary**

➪ **Compare brute-force sorting**

➪ **Assignment vs. the other two**

Known better efficiency

➪ **Solving TSP and Knapsack**

Quiz
What would be an exhaustive search solution to sorting? Compare to selection sort.

Exercise
Although can be solved the same way, why is the assignment problem fundamentally different? Check textbook for answer.

📖 Ch 11-12